University of California, Berkeley
Physics H7B Spring 1999 (*Strovink*)

SOLUTION TO PROBLEM SET 3

1. RHK problem 25.27
Solution: In Physics H7B, all problems involving numbers should be solved completely in terms of algebraic symbols before any numbers are plugged in (otherwise it is much more difficult to give part credit). Let
Q = (unknown) heat added to gas (J)
n = no. of moles of gas = 4.34
C_p = molar specific heat of gas at constant pressure (J/mole·K)
ΔT = change in temperature of gas = 62.4 K
R = universal gas constant = 8.314 J/mole·K
E_{int} = internal energy of gas (J)
M = molecular weight of gas (kg/mole)
$\langle v^2 \rangle$ = mean square velocity of gas molecules (m²/sec²)
(a)
$$Q = nC_p\Delta T \quad \text{(RHK Eq. 25.17)}$$
$$C_p = \frac{7}{2}R \quad \text{(RHK Eq. 25.21)}$$
$$Q = \frac{7}{2}nR\Delta T$$
$$= 7880 \text{ J}.$$

(b)

$$E_{int} = \frac{5}{2}nRT \quad \text{(RHK Eq. 23.36)}$$
$$\Delta E_{int} = \frac{5}{2}nR\Delta T$$
$$= 5629 \text{ J}.$$

(c)

$$n\left(\frac{1}{2}M\langle v^2\rangle\right) = \frac{3}{2}nRT \quad \text{(RHK Eq. 23.31)}$$
$$n\left(\frac{1}{2}M\Delta\langle v^2\rangle\right) = \frac{3}{2}nR\Delta T$$
$$= 3377 \text{ J}.$$

2. RHK problem 25.34
Solution: Plunging blindly ahead, we could start by assuming that "*quickly*" means quickly enough so that a negligible amount of heat is transferred between the gas and the ice water, but slowly enough to allow the pressure nevertheless to be defined (as it is in RHK Fig. 25b); and that "*slowly*" means slowly enough that the gas and the ice water always have the same temperature. If so, the "quick" compression of the gas would occur along an adiabat, while the "slow" expansion would occur along an isotherm. Then

$$W = -\int_{V_1}^{V_2} p\,dV - \int_{V_2}^{V_2} p\,dV - \int_{V_2}^{V_1} p\,dV .$$

Further assuming that the gas is ideal,

$$pV^\gamma = p_1V_1^\gamma \text{ (adiabat)}$$
$$pV = p_1V_1 \text{ (isotherm)}$$

$$W = -\int_{V_1}^{V_2} \frac{p_1V_1^\gamma}{V^\gamma}\,dV - 0 - \int_{V_2}^{V_1} \frac{p_1V_1}{V}\,dV$$

$$= -p_1V_1^\gamma \frac{-1}{\gamma-1}\left(\frac{1}{V_2^{\gamma-1}} - \frac{1}{V_1^{\gamma-1}}\right) - p_1V_1\ln\frac{V_1}{V_2}$$

$$= \frac{p_1V_1}{\gamma-1}\left(\frac{V_1^{\gamma-1}}{V_2^{\gamma-1}} - 1\right) - p_1V_1\ln\frac{V_1}{V_2} .$$

The above is correct, given the assumptions, but it does not solve the problem; we are supplied neither the initial volume nor the number of moles of gas. Instead we are told that $m = 0.122$ kg of ice in the surrounding ice water are melted in one cycle. The heat $-Q = L_f m$ required to melt this ice, where $L_f = 333$ kJ/kg is the latent heat of fusion of water, must be transferred *from* the gas (we call it $-Q$ because $+Q$ is defined to be the heat transferred *to* the gas). Around one cycle, the final temperature of gas is the same as the initial; its internal energy, which depends only on the temperature, can undergo no net change. Therefore, around the cycle, the work W done *on* the gas is given without any assumptions by

$$\Delta E_{int} = 0$$
$$\Delta E_{int} = Q + W \quad (1^{st} \text{ Law})$$
$$W = -Q$$
$$= L_f m$$
$$= 40626 \text{ J}.$$

3. RHK problem 25.37
Solution:
(a)

$$W = -\int_{V_1}^{V_2} p\, dV$$

$$\frac{V}{V_1} = \frac{p}{p_1} \quad \text{(from problem)}$$

$$W = -\int_{V_1}^{V_2} \frac{p_1}{V_1} V\, dV$$

$$= -\frac{1}{2}\frac{p_1}{V_1}(V_2^2 - V_1^2)$$

$$= -\frac{1}{2}p_1 V_1 \left(\left(\frac{V_2}{V_1}\right)^2 - 1 \right)$$

$$p_1 V_1 = nRT_1$$

$$W = -\frac{1}{2}nRT_1 \left(\left(\frac{V_2}{V_1}\right)^2 - 1 \right)$$

$$V_2 = 2V_1$$

$$W = -\frac{1}{2}nRT_1(4 - 1)$$

$$W = -\frac{3}{2}nRT_1 \ .$$

(b)

$$E_{\text{int}} = \frac{3}{2}nRT \quad \text{(ideal monatomic gas)}$$

$$\Delta E_{\text{int}} = \frac{3}{2}nR(T_2 - T_1)$$

$$nRT_2 = p_2 V_2$$

$$= (2p_1)(2V_1)$$

$$= 4p_1 V_1$$

$$= 4nRT_1$$

$$T_2 - T_1 = 3T_1$$

$$\Delta E_{\text{int}} = \frac{3}{2}nR(3T_1)$$

$$= \frac{9}{2}nRT_1 \ .$$

(c)

$$\Delta E_{\text{int}} = Q + W \quad (1^{\text{st}} \text{ Law})$$

$$Q = \Delta E_{\text{int}} - W$$

$$= \frac{9}{2}nRT_1 - \left(-\frac{3}{2}nRT_1\right)$$

$$= 6nRT_1 \ .$$

(d)

$$C \equiv \frac{Q}{n\Delta T} \quad \text{(RHK Eq. 25.8)}$$

$$= \frac{6nRT_1}{n(4T_1 - T_1)}$$

$$= 2R \ .$$

4. RHK problem 25.43
Solution: This problem is "overconstrained": that is, too many pieces of information are provided. For example, T_C need not have been supplied; it is uniquely determined by the facts that process BC is adiabatic; that $V_B = V_A$; that $p_C = p_A$; and that the gas is ideal monatomic. This is illustrated by the following calculation (not required as part of the solution):

$$p_B V_B^\gamma = p_C V_C^\gamma \quad \text{(adiabatic)}$$

$$p_B V_A^\gamma = p_A V_C^\gamma$$

$$pV = nRT$$

$$p_B = p_A \frac{T_B}{T_A}$$

$$V_A^\gamma = \left(\frac{nRT_A}{p_A}\right)^\gamma$$

$$V_C^\gamma = \left(\frac{nRT_C}{p_A}\right)^\gamma$$

$$p_A \frac{T_B}{T_A}\left(\frac{nRT_A}{p_A}\right)^\gamma = p_A\left(\frac{nRT_C}{p_A}\right)^\gamma$$

$$\frac{T_B}{T_A}T_A^\gamma = T_C^\gamma$$

$$T_C = T_A\left(\frac{T_B}{T_A}\right)^{1/\gamma}$$

$$= 454.71497 \text{ K} \ .$$

If we needed to get exact answers, we would need to plug in this exact value of T_C, rather than the approximate value of 455 K supplied in the problem. To proceed further, we choose not to use one known piece of information – not to use (as we did above) the specific relationship between p, V, T, and γ for an adiabatic transition. Because this choice is subjective and not unique, when our solutions are expressed in algebraic symbols we expect them also not to be unique. However, as long as the exact value of T_C is plugged in, we expect any valid solution to yield the same numerical results.
Let

T_A = temperature at point A = 300 K
T_B = temperature at point B = 600 K
T_C = temperature at point C = 454.71497 K
(see above discussion)
n = no. of moles of monatomic ideal gas = 1.00
R = universal gas constant = 8.314 J/mole·K
$p_A = 1.013 \times 10^5$ Pa.
Then
(a)
Process AB:

$$\Delta E_{\text{int}} = \frac{3}{2}nR(T_B - T_A)$$
$$= 3741 \text{ J} .$$
$$W = -\int_{V_A}^{V_B} p\, dV$$
$$= -\int_{V_A}^{V_A} p\, dV$$
$$= 0 .$$
$$Q = \Delta E_{\text{int}} - W$$
$$= \frac{3}{2}nR(T_B - T_A) \ (= C_V \Delta T)$$
$$= 3741 \text{ J} .$$

Process BC:

$$\Delta E_{\text{int}} = \frac{3}{2}nR(T_C - T_B)$$
$$= -\frac{3}{2}nR(T_B - T_C)$$
$$= -1812 \text{ J} .$$
$$Q = 0 \ \text{(adiabatic)} .$$
$$W = \Delta E_{\text{int}} - Q$$
$$= -\frac{3}{2}nR(T_B - T_C)$$
$$= -1812 \text{ J} .$$

Process CA:

$$\Delta E_{\text{int}} = \frac{3}{2}nR(T_A - T_C)$$
$$= -\frac{3}{2}nR(T_C - T_A)$$
$$= -1929.45 \text{ J} .$$
$$W = -\int_{V_C}^{V_A} p\, dV$$
$$= -pV_A + pV_C \ \ (p = p_A = p_C)$$
$$pV = nRT$$
$$W = -nRT_A + nRT_C$$
$$= nR(T_C - T_A)$$
$$= 1286.30 \text{ J} .$$
$$Q = \Delta E_{\text{int}} - W$$
$$= -\frac{3}{2}nR(T_C - T_A) - nR(T_C - T_A)$$
$$= -\frac{5}{2}nR(T_C - T_A) \ \ (= C_p \Delta T)$$
$$= -3215.75 \text{ J} .$$

Complete cycle:

$$\Delta E_{\text{int}} \equiv 0 \ \text{(state variable)} .$$
$$W = -\frac{3}{2}nR(T_B - T_C) + nR(T_C - T_A)$$
$$= -nRT_A - \frac{3}{2}nRT_B + \frac{5}{2}nRT_C)$$
$$= -525.55 \text{ J} .$$
$$Q = \frac{3}{2}nR(T_B - T_A) - \frac{5}{2}nR(T_C - T_A)$$
$$= nRT_A + \frac{3}{2}nRT_B - \frac{5}{2}nRT_C$$
$$= 525.55 \text{ J} .$$

(b)

$$\frac{p_B}{p_A} = \frac{T_B}{T_A} \quad (V \text{ fixed})$$

$$p_B = p_A \frac{T_B}{T_A}$$

$$= 2.026 \times 10^5 \text{ Pa} .$$

$$p_C = p_A$$

$$= 1.013 \times 10^5 \text{ Pa} .$$

$$V_A = \frac{nRT_A}{p_A}$$

$$V_B = V_A$$

$$= \frac{nRT_A}{p_A}$$

$$= 0.0246 \text{ m}^3 .$$

$$\frac{V_C}{V_A} = \frac{T_C}{T_A} \quad (p \text{ fixed})$$

$$V_C = V_A \frac{T_C}{T_A}$$

$$= 0.0373 \text{ m}^3 .$$

5. RHK problem 26.16
Solution: Consider a Carnot engine operating in reverse (as a refrigerator) between a cold reservoir at temperature $T_L = 276$ K and a hot reservoir at $T_H = 308$ K. Like all Carnot engines it is characterized by the equality

$$\frac{|Q_H|}{|Q_L|} = \frac{T_H}{T_L} \quad (\text{RHK Eq. 26.9}) .$$

For operation as a refrigerator, the heat Q_H added to the gas by the hot reservoir is negative. Conversely, Q_L is positive. The net heat $Q = Q_H + Q_L$ added to the gas over one complete cycle is negative. Since the internal energy E_{int} is a state function, over a complete cycle it must be conserved. Therefore, in one complete cycle, $-Q$ must be balanced by the mechanical work W done on the gas. A figure of merit $\mathcal{F}$ for a heat pump, the ratio of $-Q_H$ to W, is

$$\mathcal{F} = \frac{-Q_H}{W}$$

$$= \frac{-Q_H}{-Q_H - Q_L}$$

$$= \frac{T_H}{T_H - T_L}$$

$$= 9.625 .$$

The inventor claims to have achieved a figure of merit equal to

$$\mathcal{F} = \frac{-Q_H}{W}$$

$$= \frac{20 \text{ kW}}{1.9 \text{ kW}}$$

$$= 10.526 .$$

This is slightly larger than the Carnot figure of merit. Any reversible heat pump will have the same figure of merit as a Carnot engine. The only other possibility would be that the inventor's heat pump is irreversible. For example, friction in the refrigerator could convert a certain additional amount W' of work directly to heat in each cycle. In the best case, all of the heat from W' would be dumped into the hot rather than the cold reservoir. Then W' would be added both to the numerator and to the denominator of $\mathcal{F}$, reducing its physical value further below the value reported by the inventor. Therefore we are forced to reject the inventor's claim. (Nevertheless, many patents indeed have been granted for processes that violate elementary physical laws.)

6. RHK problem 26.19
Solution: Again a Carnot engine is operated in reverse between a hot reservoir at T_H and a cold reservoir at T_L. Again Q_H is negative and Q_L is positive, and, since the refrigerator is reversible,

$$\frac{|Q_H|}{|Q_L|} = \frac{T_H}{T_L} .$$

Again ΔE_{int} must be zero over a complete cycle, so that $W = -Q$ over the cycle.
(a)

$$W = -Q$$

$$= -Q_H - Q_L$$

$$= Q_L \left(\frac{-Q_H}{Q_L} - 1 \right)$$

$$= Q_L \left(\frac{T_H}{T_L} - 1 \right)$$

$$= Q_L \frac{T_H - T_L}{T_L} .$$

(b)

$$\mathcal{K} \equiv \frac{Q_L}{W}$$
$$= \frac{Q_L}{-Q_H - Q_L}$$
$$= \frac{1}{\frac{-Q_H}{Q_L} - 1}$$
$$= \frac{1}{\frac{T_H}{T_L} - 1}$$
$$= \frac{T_L}{T_H - T_L} \ .$$

(c)

$$T_L = 260 \text{ K}$$
$$T_H = 298 \text{ K}$$
$$\mathcal{K} = \frac{T_L}{T_H - T_L}$$
$$= 6.842 \ .$$

7. RHK problem 26.23
Solution: Let
Q_1 = heat transferred to gas in engine from (hot) reservoir 1 (> 0)
Q_2 = heat transferred to gas in engine from (cold) reservoir 1 (< 0)
Q_1 = heat transferred to gas in refrigerator from (hot) reservoir 3 (< 0)
Q_1 = heat transferred to gas in engine from (cold) reservoir 4 (> 0)
W_E = mechanical work done on gas in engine (< 0)
W_R = mechanical work done on gas in refrigerator (> 0)

Then
$$-W_E = Q_1 + Q_2$$
$$= Q_1\left(1 - \frac{-Q_2}{Q_1}\right)$$
$$= Q_1\left(1 - \frac{T_2}{T_1}\right)$$
$$W_R = -Q_3 - Q_4$$
$$= -Q_3\left(1 - \frac{Q_4}{-Q_3}\right)$$
$$= -Q_3\left(1 - \frac{T_4}{T_3}\right)$$
$$1 = \frac{-W_R}{W_E}$$
$$= \frac{-Q_3\left(1 - \frac{T_4}{T_3}\right)}{Q_1\left(1 - \frac{T_2}{T_1}\right)}$$
$$\frac{-Q_3}{Q_1} = \frac{1 - \frac{T_2}{T_1}}{1 - \frac{T_4}{T_3}}$$
$$\frac{|Q_3|}{|Q_1|} = \frac{1 - \frac{T_2}{T_1}}{1 - \frac{T_4}{T_3}} \ .$$

8. RHK problem 26.27
Solution: Let
W_{ab} = work done on gas in stroke ab, etc.
W = work done *on* gas in cycle
$W_{\text{by eng}}$ = work done *by* engine in cycle
p_0 = smaller pressure = 1.01×10^5 Pa
p_1 = larger pressure = $2p_0$
V_0 = smaller volume = 0.0225 m^3
V_1 = larger volume = $2V_0$
Q_{abc} = heat added to gas during pair of expansion strokes
e = efficiency of engine
e_{Carnot} = efficiency of Carnot engine operating between two temperatures with ratio $p_1 V_1 / p_0 V_0$
Then
(a)

$$W = W_{ab} + W_{bc} + W_{cd} + W_{da}$$
$$= 0 - p_1(V_1 - V_0) + 0 + p_0(V_1 - V_0)$$
$$= -(p_1 - p_0)(V_1 - V_0)$$
$$W_{\text{by eng}} = -W$$
$$= (p_1 - p_0)(V_1 - V_0)$$
$$= p_0 V_0$$
$$= 22725 \text{ J} \ .$$

(b)

$$Q_{abc} = Q_{ab} + Q_{bc}$$
$$= E_{\text{int}}(c) - E_{\text{int}}(a) - W_{ab} - W_{bc}$$
$$= \frac{3}{2}nR(T_c - T_a) - 0 + p_1(V_1 - V_0)$$
$$= \frac{3}{2}p_1V_1 - \frac{3}{2}p_0V_0 + p_1V_1 - p_1V_0$$
$$= \frac{5}{2}p_1V_1 - p_1V_0 - \frac{3}{2}p_0V_0$$
$$= \frac{13}{2}p_0V_0$$
$$= 147713 \text{ J} \ .$$

(c)

$$e \equiv \frac{W_{\text{by eng}}}{Q_{abc}}$$
$$= \frac{(p_1 - p_0)(V_1 - V_0)}{\frac{5}{2}p_1V_1 - p_1V_0 - \frac{3}{2}p_0V_0}$$
$$= \frac{2}{13}$$
$$= 0.1538 \ .$$

(d)

$$e_{\text{Carnot}} = \frac{T_c - T_a}{T_c}$$
$$= \frac{p_1V_1 - p_0V_0}{p_1V_1}$$
$$= \frac{3}{4} \ .$$